Quantum-classical comparison in chaotic systems

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We introduce a characteristic time of a classical chaotic dynamics, represented by the coherence time of the local maximum expansion direction. For a quantum system whose classical limit follows the above chaotic dynamics, the ratio between this time and the decorrelation time (of the order of the reciprocal of the maximum Liapunov exponent) rules the ratio between nonclassical (Moyal) and classical (Liouville) terms in the evolution of the density matrix. We show that such a ratio does not provide a complete criterion for quantum-classical correspondence.

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The problem of the correspondence between the classical and the quantum evolution of a Hamiltonian system has been recently considered [1–7] in various situations. A quantum-classical correspondence (QCC) implies an answer to the question of the limitations imposed by the quantum nature of the system to the measuring process [8].

In particular, recent papers [9] have shown that the ratio between nonclassical and classical terms in the evolution equation of the phase space density diverges for an unstable motion, whereas it decays to zero if one accounts for a coupling with the environment, and that decay was taken as a definition of QCC.

In this paper we present a relevant case whereby (i) that ratio remains confined to very small values even for an isolated system (i.e., in the absence of the environment), because of the intrinsic spread of the chaotic motion, and yet (ii) a QCC, defined more rigorously as the absence of appreciable differences between classical and quantum phase space densities, is not achieved, since for long times the quantum phase space density shows appreciable deviations from the classical one. Claim (i) is based on the introduction of a chaotic indicator, not considered previously in classical chaos; claim (ii) is supported by numerical evidence. Precisely, we refer to a classically chaotic nonautonomous system (i.e., with a time dependent forcing term), which models a Hydrogen atom in a Rydberg state excited by a microwave field [10].

The quantum evolution is described by the quasiprobability Wigner function [11]

$$W(q,p)=\frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}dy e^{(i/\hbar)py}\left\langle q-\frac{y}{2}|\hat{\rho}|q+\frac{y}{2}\right\rangle, \quad (1)$$

where $\hat{\rho}$ is the density operator of the system, p and q are the conjugate variables and $\langle \ldots \rangle$ stays for the expectation value.

The Wigner function can take negative values, thus it is not a probability function [11]; however, it represents a good tool for inspecting the classical or quantum nature of the system. Indeed, its time evolution is ruled by

$$\dot{W}=\{H,W\}_{\mathrm{PB}}+\sum_{n\geqslant 1}\frac{\hbar^{2n}(-1)^n}{2^{2n}(2n+1)!}\partial_x^{2n+1}V\partial_p^{2n+1}W, \quad (2)$$

where H is the Hamiltonian of the system written in the Wigner form [12], V is the potential, $\{\ \}_{\mathrm{PB}}$ stays for the Poisson brackets [13], and ∂_x,∂_p stay for $\partial/(\partial x),\partial/(\partial p)$. Notice that the right hand side of Eq. (2) contains two different terms which contribute to the evolution of W. The Poisson bracket term generates the ordinary Liouville flow, which corresponds to the classical W evolution, while the sum represents the contribution of the Moyal terms [14] to the quantum evolution of W.

In Ref. [9] the Moyal terms become comparable to the Poisson bracket at a time τ_1 proportional to the reciprocal of the Liapunov exponent Λ. There, the growth of the Moyal contributions is quenched by the coupling with a thermal bath, which involves further diffusive and dissipative terms to be added to Eq. (2). Even though dissipation and diffusion guarantee the vanishing of the Moyal terms on the long time scales and thus the correspondence between the quantum and classical description, here we are interested in finding sufficient conditions for confining the Moyal contributions within a limited range for isolated systems.

Let us introduce one of the simplest chaotic dynamics which gives rise to nonzero Moyal contributions. The Hamiltonian of the system is

$$H=\frac{p^2}{2m}+V(x,t), \quad (3)$$

where m is the mass of the system, $V(x,t)=V_1(x)+V_2(x,t)$, $V_1=-(1/2)\alpha x^2+(1/4)\beta x^4$, and $V_2=\epsilon x\ \cos\omega t$.

Here V_1 is the unperturbed potential, V_2 is the driving potential, and $\alpha,\beta,\epsilon,\omega$ are the real parameters to be later specified. Equation (2) becomes

$$\dot{W}=(-\alpha x+\beta x^3+\epsilon\ \cos\omega t)\frac{\partial W}{\partial p}-\frac{p}{m}\frac{\partial W}{\partial x}-\frac{\hbar^2}{4}\beta x\frac{\partial^3 W}{\partial p^3}, \quad (4)$$

while the classical behavior is ruled by the Hamilton-Jacobi equations which read as

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$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m},$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -\alpha x + \beta x^3 + \epsilon \cos\omega t. \tag{5}$$

Equation (5) can give rise to a chaotic dynamics. Furthermore, choosing $\omega \gg (\alpha/m)^{1/2}$ (forcing frequency much higher than the proper frequency of the autonomous part) preserves the conservative nature (in average) of the system.

The guiding line of our approach is that, for a bounded chaotic system, another characteristic time scale arises in addition to τ_1, due to the continuous stretching and folding process which causes a twisting of the direction of the eigenvector corresponding to the local maximum Liapunov exponent λ. As a consequence, the system does not always expand in the same direction of phase space, but, on the contrary, each variable experiences both contraction and expansion depending upon the local position on the trajectory.

Thus besides the decorrelation time τ_1, one must introduce another time scale τ_2, proportional to the reciprocal of the averaged frequency $\langle\Omega\rangle$ of the twisting of the direction locally singled by the eigenvector corresponding to λ.

Depending on the values of τ_1 and τ_2, we classify the chaotic dynamics as follows. In those systems for which τ_1 is smaller than, or of the order of τ_2, the direction of λ remains invariant over the time scale sufficient for the Moyal contributions to be comparable with the Liouville terms. Hence these systems are intrinsically quantum. A transition from the quantum to the classical description can be achieved only through the coupling to an environment [9].

On the contrary, if $\tau_1 \gg \tau_2$, the growth of the Moyal terms on a single decorrelation time is limited by the change of the direction of the maximum expansion. Indeed, the growth of the Moyal terms is due to the fact that, since they are proportional to the powers of the reciprocal of $\Delta x(\Delta p)$, if one of the two conjugate variables is locally expanding, the other one is contracting to preserve the volume conservation in the phase space. As a consequence, those terms containing powers of the contracting variable in the denominator tend to explode. To apply these general considerations to the specific chaotic dynamics, we first identify the range of the control parameters for which the system belongs to the former or to the latter case.

For the classical evolution of the driven double-well system [Eq. (5)], Fig. 1 shows the plot of Λ and $\langle\Omega\rangle$ against the control parameter ϵ. Λ has been calculated with the algorithm by Benettin *et al.* [15]. As for $\langle\Omega\rangle$, we have introduced the following strategy. At each time t_n we consider a circle of radius η centered at the intersection of the actual trajectory with a plane transversal to the flow. In our case, for any time, this plane coincides with the two dimensional (x,p) projection of the phase space. We let all the points on the circle evolve to the time $t_{n+1}(t_{n+1}-t_n=\tau_{\mathrm{RK}}$, where τ_{RK} is the Runge-Kutta integration interval). At this new time the points will form an ellipse about the time evolved of the actual trajectory. The direction of the local eigenvector corresponding to λ coincides with the direction of the major axis of such an ellipse.

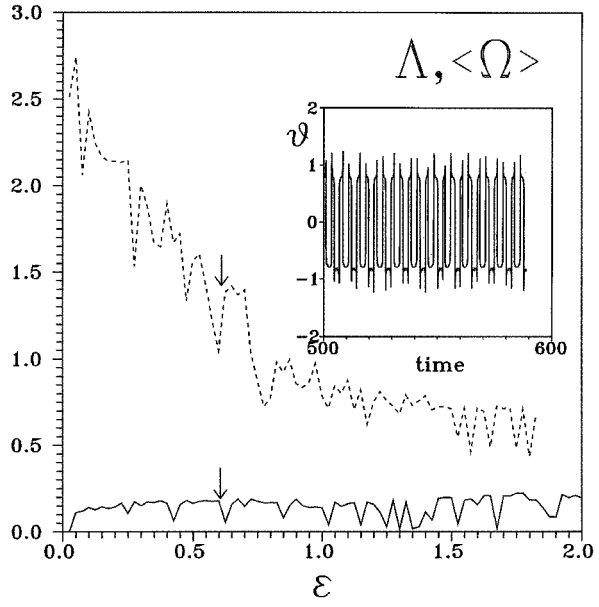

FIG. 1. Maximum Liapunov exponent Λ (solid line) and averaged twisting frequency $\langle\Omega\rangle$ (dashed line) as functions of the forcing amplitude ϵ for the driven double-well system. Λ and $\langle\Omega\rangle$ have been evaluated for the classical system [Eq. (5)] with $\alpha=1$, $\beta=2$, $m=1$,$\omega=2.5$. Arrows indicate a possible choice for which $\langle\Omega\rangle \gg \Lambda$. Inset: Temporal evolution of the angle θ formed by the eigenvector corresponding to the local maximum expansion direction with respect to the x axis. θ has been calculated for $\epsilon=0.7$ and the above values of the other parameters.

For a given ϵ we have reported in the inset of Fig. 1 the temporal evolution of the angle θ formed by the major axis of the local ellipse with the x axis. The twisting process determines a characteristic oscillation of θ, the averaged frequency of which is $\langle\Omega\rangle$. Looking at Fig. 1 we can easily locate the ranges of the control parameter ϵ for which $\tau_1 \gg \tau_2$ holds.

Moving now to the quantum system ruled by the same Hamiltonian, we will focus on the $\tau_1 \gg \tau_2$ case to show that the Moyal terms are limited and therefore we cannot invoke their growth to explain the lack of the QCC.

For this purpose, Eq. (4) has been numerically integrated over a $N \times N$ two dimensional (x,p) grid, where the maximum and minimum x and p values have been selected accordingly to the choice of α and β. At each time t_n, the Liouville terms have been evaluated with the Lax-Wendroff scheme [16] due to the hyperbolic nature of such terms, while the Moyal contributions have been evaluated using a finite difference method for the third order partial derivatives [17]. Boundary conditions implying the setting to zero of the derivatives at the borders of the grid have been chosen. The numerical integration has been performed over a global time much larger than $1/\omega$ in order to assure the conservative nature of the quantum solution. A useful parameter is also the log time $t_{\log}$ [18]

$$t_{\log} = \frac{1}{\Lambda} \ln\left(\frac{S}{\hbar}\right). \tag{6}$$

In our case ($\hbar = 1/200\pi$ [5]), taking S to be the action per bounce in the double-well potential at the threshold orbit, the

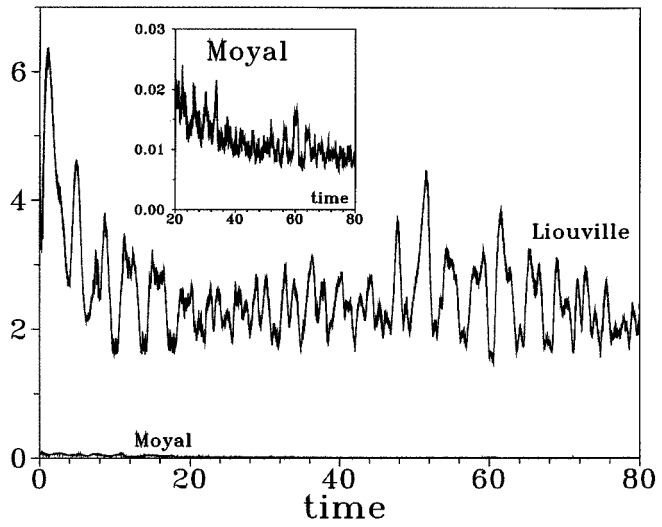

FIG. 2. Plot of the maxima of the Liouville and Moyal terms in the course of time for $\epsilon=0.7, \hbar=1/200\pi$ and other parameters as in Fig. 1. A Gaussian wave packet centered in $x=0.3$ and $p=0.1$ whose Wigner function is $W(q,p)=(1/\pi\hbar)e^{-(x-x_0)^2/\delta^2-[(p-p_0)^2\delta^2/\hbar^2]}$ ($\delta=0.1$) has been chosen as an initial condition. Boundary conditions as discussed in the text. Inset: vertically magnified plot of Moyal contributions. Notice that the windows confining the Moyal term remains limited, even for $t>\tau_1\approx(1/\Lambda)\ln(S/\hbar)\sim 50$ sec.

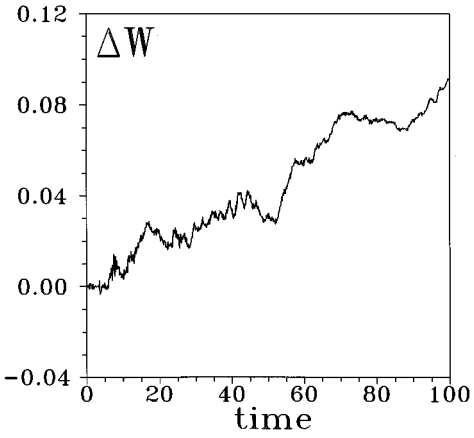

FIG. 3. Differences ΔW (see text) between quantum and classical integrated Wigner functions extracted from the solution of Eq. (4) with $\epsilon=0.7$, $\hbar=1/200\pi$, and 0, respectively. Same boundary and initial conditions as in Fig. 2.

argument of the logarithm is of the order 200–300. So, for $1/\Lambda=10$ (see Fig. 1), this gives $t_{\log}=50-60$. By Berry's argument [18], integration over a time much longer than $t_{\log}$ should yield appreciable differences between the quantum and the classical system.

On the other hand, calling M (L) the Moyal (Liouville) terms of the evolution equation (4), for $\tau_1 \gg \tau_2$, that is, for $\Lambda \ll \langle\Omega\rangle$, one should expect that M (L) be confined within an amplitude range $M_0 e^{3(|\lambda|/\langle\Omega\rangle)}$ ($L_0 e^{|\lambda|/\langle\Omega\rangle}$), M_0 (L_0) being the initial amplitude. This is due to the fact that expansion of M (L) due to the contraction of Δx or Δp lasts only for a time $\tau_2=1/\langle\Omega\rangle$. Thus the ratio

$$\frac{|M|}{|L|}\sim\frac{|M_0|}{|L_0|}e^{2(\Lambda/\langle\Omega\rangle)} \tag{7}$$

can be limited by a suitable choice of the ratio $\Lambda/\langle\Omega\rangle$. Notice that the ratio $|M_0|/|L_0|$ can be arbitrarily chosen by selecting a suitable classical initial state. Equation (7) shows that for $\Lambda \ll \langle\Omega\rangle$ the ratio of the Moyal to the Liouville terms can be adjusted to a preassigned value. This limitation is guaranteed for all $\hbar$ values which give rise to negligible $|M_0|/|L_0|$ ratios.

Figure 2 is a plot of the maxima of the Moyal and the Liouville terms in the course of time. As shown by the magnified inset, the Moyal contributions are always much smaller than the Liouville terms despite the absence of the diffusive and dissipative terms in Eq. (4), at variance with the expectation of Ref. [9]. Notice that this limitation is obtained in the range of ϵ values (extracted from Fig. 1) for which the classical chaotic loss of directional coherence of the maximum expansion is much faster than the decorrelation time.

Yet the relative smallness of the Moyal terms does not necessarily imply a small dynamical effect, thus, it is not a sufficient condition for a QCC defined as the ''absence of appreciable differences between classical and quantum phase space densities.''

To compare the quantum and classical systems we have calculated at each integration time the differences, point by point, between the quantum Wigner function W_Q obtained by integration of the full Eq. (4) and the classical Wigner function W_C obtained from Eq. (4) without the Moyal term, $\delta W(x,p,t)=W_Q(x,p,t)-W_C(x,p,t)$.

We integrate δW over the grid, obtaining $\int_{\text{all the grid}}\delta W(x,p,t)dx\,dp$ and normalize to the grid integral of the Wigner distribution. Calling ΔW such a ratio, we plot in Fig. 3 its temporal evolution. If we call M the average of M over the grid, then ΔW is proportional to $\int \overline{M}dt$.

Due to the fact that $\langle\Omega\rangle/\Lambda>1$, the maxima of the Moyal term ($M_{\max}$) remain limited over a time longer than any relevant time of the system. Thus ΔW has an upper limit smaller than $M_{\max}\cdot t$. Indeed, the temporal evolution of ΔW is approximately fitted by a linear growth (Fig. 3).

For more complex systems, even though $\langle\Omega\rangle$ is still a relevant indicator, the direction of λ generally does not lie on the (x,p) plane. Our argument, having to combine dynamical aspects with the quantum conjugacy of the two variables x and p, is limited to a single degree of freedom (x,p) with a forcing term. However, it is already descriptive of a large test class for quantum chaos [6] [10].

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